COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA 17 (4) (1997) 523–554

FINITE GROUP ACTIONS AND ASYMPTOTIC EXPANSION OF $e^{P(z)}$

THOMAS MÜLLER[1]

Received November 18, 1996

We establish an asymptotic expansion for the number $|\mathrm{Hom}\,(G,S_n)|$ of actions of a finite group G on an n-set in terms of the order $|G|=m$ and the number $s_G(d)$ of subgroups of index d in G for $d\mid m$. This expansion and related results on the enumeration of finite group actions follow from more general results concerning the asymptotic behaviour of the coefficients of entire functions of finite genus with finitely many zeros. As another application of these analytic considerations we establish an asymptotic property of the Hermite polynomials, leading to the explicit determination of the coefficients $C_\nu(\alpha;z)$ in Perron's asymptotic expansion for Laguerre polynomials in the cases $\alpha=\pm 1/2$.

1. Introduction

The principal aim of the present paper is to investigate the problem of asymptotically enumerating finite group actions. More precisely, we will be concerned with the following question.

(1) *Given a finite group G, how does the number $|\mathrm{Hom}\,(G,S_n)|$ of G-actions on an n-set behave as $n\to\infty$?*

This combinatorial problem has, at least in the case when G is cyclic, received a fair amount of attention since the early 1950's. Among the earliest contributions are two papers [2] and [3] by Chowla et al. In [2] Chowla, Herstein and Moore consider the sequence $T_n:=|\mathrm{Hom}\,(C_2,S_n)|$. They find the exponential generating function and some divisibility properties of the T_n and establish the asymptotic formula

$$T_n \sim 2^{-1/2} n^{n/2} \exp\left(-\frac{n}{2}+n^{1/2}-\frac{1}{4}\right) \quad (n\to\infty) \tag{2}$$

using methods from real analysis. In [3] Chowla, Herstein and Scott consider, more generally, the function $|\mathrm{Hom}\,(C_m,S_n)|$ counting the solutions of the equation $X^m=1$ in the symmetric group S_n for each fixed m. They find the generating function and some recurrence relations, and explicitly raise the problem of finding an asymptotic formula for $|\mathrm{Hom}\,(C_m,S_n)|$ with m fixed and $n\to\infty$. An important

Mathematics Subject Classification (1991): 05A16, 41A60, 05E99, 20F99

[1] Research supported by Deutsche Forschungsgemeinschaft through a Heisenberg-Fellowship.

step was taken in 1955 when Moser and Wyman reproved (a somewhat refined version of) formula (2) using methods from complex analysis. In the same paper [11] they also verified a conjecture concerning the asymptotic behaviour of the quotient T_n/T_{n-1} made in [2] (we shall come back to the latter point in Section 5). In a subsequent paper [12] they further improved the method of [11, Sect. 3] and, consequently, were able to handle the case of groups of prime order. Their result is that

$$(3) \qquad |\mathrm{Hom}\,(C_p, S_n)| \sim K_p n^{(1-1/p)n} \exp\left(-\frac{p-1}{p}n + n^{1/p}\right) \quad (n \to \infty),$$

where $K_p = p^{-1/2}$ for primes $p>2$ and $K_2 = 2^{-1/2}e^{-1/4}$. More recently, Wilf [16] has proved that for fixed m, as $n \to \infty$

$$|\mathrm{Hom}\,(C_m, S_n)|/(n!) \sim \frac{\tau^n}{\sqrt{2\pi mn}} \exp\left(\sum_{d|m} \frac{1}{d\tau^d}\right),$$

where

$$\tau = \tau(m,n) := n^{-1/m}\left\{1 + \frac{1}{mn}\sum_{\substack{d|m\\ d<m}} n^{d/m} + \varepsilon_{m,n}\right\}$$

and

$$\varepsilon_{m,n} := \begin{cases} 1/(2m^2n); & m \text{ even} \\ 0; & m \text{ odd.} \end{cases}$$

Here, we will establish a complete asymptotic expansion of the function $|\mathrm{Hom}(G,S_n)|$ for an arbitrary finite group G in terms of the order $|G|=m$ and the number $s_G(d)$ of subgroups of index d in G for $d|m$; cf. Section 5, Theorem 5. Its main term yields the asymptotic formula

$$(4) \quad |\mathrm{Hom}\,(G, S_n)| \sim K_G n^{(1-1/m)n} \exp\left(-\frac{m-1}{m}n + \sum_{\substack{d|m\\ d<m}} \frac{s_G(d)}{d} n^{d/m}\right) \quad (n \to \infty)$$

with

$$K_G := \begin{cases} m^{-1/2}; & m \text{ odd} \\ m^{-1/2}\exp\left(-\frac{(s_G(m/2))^2}{2m}\right); & m \text{ even,} \end{cases}$$

and the quotient

$$|\mathrm{Hom}\,(G, S_n)| \Big/ \left(K_G n^{(1-1/m)n} \exp\left(-\frac{m-1}{m}n + \sum_{\substack{d|m\\ d<m}} \frac{s_G(d)}{d} n^{d/m}\right)\right)$$

is expanded as a Poincaré series in $n^{-1/m}$ with coefficients given explicitly in terms of m and the $s_G(d)$.

By the exponential principle, the number $|\mathrm{Hom}\,(G, S_n)|$ of G-actions on an n-set is connected with the order $|G|=m$ and the subgroup numbers $s_G(d)$ via the relation

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}\,(G,S_n)|}{n!} z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d} z^d \right); \tag{5}$$

cf. for example [5, Prop. 1]. Hence, from an analytic point of view, (1) is a special case of the following.

(1′) *Given a real polynomial $P(z)$, derive asymptotic information on the coefficients α_n of the entire function $\exp(P(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$, which is explicit in $P(z)$ and n.*

Problem (1′) was studied already around 1920 by Pólya in connection with his investigation concerning the zeros of the derivatives of certain functions; cf. [13]. His result is however not sufficiently explicit for the purposes of (1). Also, Pólya's method does not give complete asymptotic expansions but only information on the first term.

In the case $P(z)=z$ a solution of (1′) is, of course, given by Stirling's formula

$$n! \sim (2\pi)^{1/2} n^{n+1/2} e^{-n} \quad (n \to \infty) \tag{6}$$

or, on a different level of precision, by the expansion

$$n! \approx (2\pi)^{1/2} n^{n+1/2} e^{-n} \left\{ 1 + \sum_{\nu=1}^{\infty} \mathfrak{c}_\nu n^{-\nu} \right\} \quad (n \to \infty) \tag{7}$$

of factorials derived from Stirling's asymptotic expansion of $\log\Gamma(z)$. As is well-known the coefficients $\mathfrak{c}_\nu$ in (7) can be expressed in terms of Bernoulli numbers via the (formal) identity

$$\exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} z^{-(2k-1)} \right) = 1 + \sum_{\nu=1}^{\infty} \mathfrak{c}_\nu z^{-\nu}. \tag{8}$$

In dealing with question (1) the crux is to obtain results for (1′) analogous to (6) or (7) for a sufficiently large class of polynomials $P(z)$ including in particular all polynomials $P_G(z) = \sum_d \frac{s_G(d)}{d} z^d$ associated with finite group actions, while maintaining this high level of explicitness in $P(z)$ and n. Indeed, interpreted in this way, (1′) is not a well-posed problem, since the class of functions $\mathcal{F} = \{e^{P(z)} : P(z) \in \mathbb{R}[z]\}$ turns out to be too large to allow for a uniform asymptotic behaviour of the coefficients α_n.

In Sections 2-4 of this paper we investigate problem (1′) and a slight extension of it under certain restrictions on $P(z)$. To be more specific, let $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu$ be a polynomial of exact degree $m \geq 1$ with real coefficients c_μ, put $\exp(P(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$, and consider the following set of conditions.

($\mathscr{P}0$) $\alpha_n > 0$ for all sufficiently large n.

($\mathscr{P}1$) $c_\mu \geq 0$ for $1 \leq \mu \leq m$.

($\mathscr{P}2$) $c_1 \neq 0$.

($\mathscr{P}3$) $c_\mu = 0$ for $m/2 < \mu < m$.

In Section 2 we derive the asymptotic formula

$$\alpha_n \sim \frac{K}{\sqrt{2\pi n}} \left(\frac{n}{mc_m}\right)^{-n/m} \exp\left(P\left(\left(\frac{n}{mc_m}\right)^{1/m}\right)\right) \quad (n \to \infty)$$

with suitable constant $K = K(P)$ under the assumption that $P(z)$ satisfies ($\mathscr{P}0$) and ($\mathscr{P}3$); cf. Theorem 1. This asymptotic formula, which readily implies (4), is extended in Theorem 2 to a complete asymptotic expansion of α_n, assuming that $P(z)$ has degree at least 2 and satisfies conditions ($\mathscr{P}1$), ($\mathscr{P}2$), and ($\mathscr{P}3$). Observe that, by a result of Schur, conditions ($\mathscr{P}1$) and

($\mathscr{P}2$)′ $\gcd \operatorname{supp}(P(z)) = 1$

imply ($\mathscr{P}0$); cf. [1]. Here, $\operatorname{supp}(P(z)) := \{\mu : c_\mu \neq 0\}$ is the support of $P(z)$. Moreover, it is not difficult to construct polynomials $P(z)$ satisfying, say, ($\mathscr{P}0$), ($\mathscr{P}2$), and ($\mathscr{P}3$), but not ($\mathscr{P}1$). For example, the polynomial

$$P_m(z) := -z + \sum_{2 \leq \mu \leq m/2} z^\mu + z^m$$

has property ($\mathscr{P}0$) for all $m \geq 5$. Hence, condition ($\mathscr{P}0$) is indeed weaker than the conjunction of ($\mathscr{P}1$) and ($\mathscr{P}2$), and Theorem 1 can be applied in situations where Theorem 2 is not available. The gap condition ($\mathscr{P}3$) has turned out to be the most efficient way of exploiting the fact that the polynomials $P_G(z)$ arising from finite group actions have the property that

$$\operatorname{supp}(P_G(z)) \subseteq \{d : d | \deg(P_G(z))\}.$$

Indeed, it is this condition which allows us to obtain completely explicit results for polynomials of arbitrary degree. In Section 4 the results of the preceding sections are extended to functions of the form $f(z) = Q(z)e^{P(z)}$ with polynomials $P(z)$ and $Q(z)$, i.e., entire functions of finite genus having only a finite number of zeros.

In Section 5 we return to question (1), collecting and commenting on what we have learned along the way concerning the problem of enumerating finite group actions; in particular, we obtain from Theorem 2 the aforementioned asymptotic expansion for the function $|\mathrm{Hom}(G, S_n)|$ attached to a finite group G (Theorem 5). We also record an asymptotic expansion of the quotient $|\mathrm{Hom}(G, S_n)|/|\mathrm{Hom}(G, S_{n-1})|$ (Theorem 6), which follows from Theorem 5 or, more directly, from Theorem 3, and, as a consequence of Theorem 1, we establish the following somewhat curious phenomenon ("asymptotic stability" of finite groups):

If for two finite groups G and H we have that

$$|\mathrm{Hom}\,(G, S_n)| \sim |\mathrm{Hom}\,(H, S_n)| \quad (n \to \infty),$$

then the two functions must in fact coincide.

As another application of Theorem 2 we establish a certain asymptotic property of the Hermite polynomials, leading to the determination of the coefficients $C_\nu(\alpha; z)$ in Perron's asymptotic expansion for Laguerre polynomials in the cases $\alpha = \pm 1/2$.

Apart from their intrinsic interest, Theorems 5 and 6 also lead to a determination of the subgroup growth for a large class of virtually free groups, including in particular all free products of the form

$$\Gamma = \mathop{*}_{\sigma=1}^{s} G_\sigma * F_r, \quad 0 \le r, s < \infty, \quad 1 < |G_\sigma| < \infty$$

and all free products with amalgamation of finite cyclic groups; cf. [9] and [10]. It was with this application in mind that I originally became interested in problem (1).

2. Asymptotic Expansion of $e^{P(z)}$: Hayman's Method

The purpose of this paragraph is to establish the following generalization of Stirling's formula.

Theorem 1. *Suppose that the polynomial* $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu \in \mathbb{R}[z]$ *has degree* $m \ge 1$ *and meets the conditions* ($\mathscr{P}0$) *and* ($\mathscr{P}3$). *Then the coefficients* α_n *of the entire function* $\exp(P(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$ *satisfy the asymptotic formula*

$$\alpha_n \sim \frac{K}{\sqrt{2\pi n}} n_0^{-n/m} \exp\left(P(n_0^{1/m})\right) \quad (n \to \infty), \tag{9}$$

where $n_0 := n/(mc_m)$ *and*

$$K = K(P) := \begin{cases} m^{-1/2}; & m \text{ odd} \\ m^{-1/2} \exp\left(-\frac{c_{m/2}^2}{8c_m}\right); & m \text{ even.} \end{cases}$$

Our starting point in proving Theorem 1 is the asymptotic formula

$$\alpha_n \sim \frac{\exp(P(r_n))}{r_n^n \sqrt{2\pi mn}} \quad (n \to \infty), \tag{10}$$

in terms of the positive real root r_n of the equation

$$rP'(r) = n, \quad n \geq n_0, \tag{11}$$

which follows from Hayman's work [6]. More specifically, the assumption that $P(z)$ has real coefficients and satisfies condition ($\mathscr{P}0$) ensures via [6, Theorem X] that the function $\exp(P(z))$ is admissible in the whole complex plane in the sense of [6, pp. 68-69], in particular in view of [6, formula (1.2)] we have $c_m > 0$. By [6, Cor. II to Theorem I] we find that

$$\alpha_n \sim \frac{\exp(P(r_n))}{r_n^n\sqrt{2\pi b(r_n)}} \quad (n \to \infty),$$

where $b(r) := rP'(r) + r^2P''(r)$. Since $c_m > 0$, r_n is well-defined and strictly increasing for sufficiently large n and unbounded as $n \to \infty$. This gives $r_n \sim n_0^{1/m}$ and $b(r_n) \sim mn$, whence (10).

In order to turn (10) into an explicit asymptotic formula we have to approximate the root r_n of equation (11) by a function ρ_n (which is explicit in n and $P(z)$) with sufficient precision to allow r_n^n and $\exp(P(r_n))$ to be estimated asymptotically also, i.e., $r_n^n \sim \rho_n^n$ and $\exp(P(r_n)) \sim \exp(P(\rho_n))$. For this we will need r_n with an (absolute) error of order $o(n^{-1+1/m})$. Now consider the function

$$\phi(z) := \left(\sum_{\mu=1}^{m} \mu c_\mu z^{m-\mu}\right)^{1/m} \underset{z\neq 0}{=} \left(z^{m-1}P'(1/z)\right)^{1/m}.$$

$\phi(z)$ is analytic in some disc centered at the origin and $\phi(0) = (mc_m)^{1/m} \neq 0$. By Lagrange's inversion theorem[2] there exist real numbers $\varepsilon > 0$ and $\delta > 0$ such that for all w with $|w| < \varepsilon$ the equation (in z) $w\phi(z) = z$ has exactly one solution in the domain $|z| < \delta$. Moreover, this solution $z(w)$ is an analytic function of w:

$$z = z(w) = \sum_{\nu=1}^{\infty} \frac{\beta_\nu}{\nu} w^\nu, \quad |w| < \varepsilon,$$

where the β_ν's are given by

$$\beta_\nu = \left\langle z^{\nu-1}, (\phi(z))^\nu \right\rangle, \quad \nu \geq 1, \tag{12}$$

i.e., β_ν is the coefficient of $z^{\nu-1}$ in the expansion of $(\phi(z))^\nu$ around the origin. Consequently, putting $w = n^{-1/m}$ and $z = r^{-1}$, we find that there exists a positive

[2] For the form of Lagrange's theorem used here see for example [4, Sect. 2.2] or [17, Sect. 7.32].

integer n_1 such that for all $n \ge n_1$ the equation $rP'(r) = n$ has exactly one (complex) solution in the domain $|r| > 1/\delta$ and that this solution r_n is of the form

$$\frac{1}{r_n} = \sum_{\nu=1}^{\infty} \frac{\beta_\nu}{\nu} n^{-\nu/m}, \quad n \ge n_1. \tag{13}$$

The right-hand side of (13) is convergent, by Lagrange's theorem, for all sufficiently large n and therefore also asymptotic as $n \to \infty$, i.e., we have

$$\frac{1}{r_n} \approx \beta_1 n^{-1/m} \left\{ 1 + \sum_{\nu=1}^{\infty} \beta_1^{-1} \frac{\beta_{\nu+1}}{\nu+1} n^{-\nu/m} \right\} \quad (n \to \infty). \tag{14}$$

Moreover, since the positive real solution r_n of the equation $rP'(r) = n$ is unbounded as $n \to \infty$ and strictly increasing from some n onward, it satisfies $|r_n| > 1/\delta$ for all sufficiently large n and, hence, must coincide for large n with the solution r_n in (13). Define a function ρ_n by

$$\rho_n := \beta_1^{-1} n^{1/m} \left\{ 1 + \sum_{\nu=1}^{m} \beta_1^{-1} \frac{\beta_{\nu+1}}{\nu+1} n^{-\nu/m} \right\}^{-1}. \tag{15}$$

Clearly, ρ_n is well-defined for sufficiently large n, and, by (14), we have in particular that $r_n = \rho_n + \mathcal{O}(n^{-1})$, and therefore

$$\alpha_n \sim \frac{\exp(P(\rho_n))}{\rho_n^n \sqrt{2\pi m n}} \quad (n \to \infty). \tag{16}$$

Note that so far we have not used assumption ($\mathcal{P}3$); however, this extra hypothesis will be needed in order to transform (16) into the simpler and more explicit form (9). By expanding $(\phi(z))^\nu = (mc_m)^{\nu/m} \left(1 + \sum_{\mu=1}^{m-1} \frac{\mu c_\mu}{mc_m} z^{m-\mu} \right)^{\nu/m}$ we immediately infer from (12) that

$$\beta_\nu = \begin{cases} (mc_m)^{1/m}; & \nu = 1 \\ \sum_{\kappa=1}^{\nu-1} \sum_{\substack{\mu_1+\ldots+\mu_\kappa = \kappa m - \nu + 1 \\ 0<\mu_1,\ldots,\mu_\kappa<m}} \binom{\nu/m}{\kappa} \mu_1 \ldots \mu_\kappa c_{\mu_1} \ldots c_{\mu_\kappa} (mc_m)^{\nu/m-\kappa}; & \nu \ge 2. \end{cases} \tag{17}$$

Consider β_ν for $\nu \ge 2$. Assuming ($\mathcal{P}3$), a summand on the right-hand side of (17) can be non-zero only if $\kappa \le 2(\nu-1)/m$. Since $\kappa \ge 1$ it follows that $\beta_\nu = 0$ for $1 < \nu < m/2 + 1$. Moreover, if $\nu = m - \lambda + 1$, $1 \le \lambda \le m/2$, the only possible contribution occurs for $\kappa = 1$ and $\mu_1 = \lambda$; in particular we find that

$$\beta_{m/2+1} = \frac{m+2}{4} c_{m/2} (mc_m)^{-\frac{m-2}{2m}}, \quad 2|m. \tag{18}$$

Abbreviating the sum $\sum_{m/2 \le \nu \le m} \beta_1^{-1} \frac{\beta_{\nu+1}}{\nu+1} n^{-\nu/m}$ as Σ and using the fact that Σ is of order $\mathcal{O}(n^{-\lceil m/2 \rceil/m})$ we find that as $n \to \infty$

$$\begin{aligned}\rho_n^{-n} &= n_0^{-n/m}(1+\Sigma)^n \\ &= n_0^{-n/m} \exp\Big(n\Sigma - n\Sigma^2/2 + o(1)\Big), \\ &= n_0^{-n/m} \exp\Big(n\Sigma - \sigma^2/2 + o(1)\Big),\end{aligned}$$

i.e.,

$$\rho_n^{-n} \sim n_0^{-n/m} \exp\Big(n\Sigma - \sigma^2/2\Big) \quad (n \to \infty), \tag{19}$$

where

$$\sigma := \begin{cases} \frac{2}{m+2}\beta_1^{-1}\beta_{m/2+1}; & m \text{ even} \\ 0; & m \text{ odd.} \end{cases}$$

In order to rewrite $\exp(P(\rho_n))$ we have to deal with the terms $\exp(\rho_n^\mu)$ for $1 \le \mu \le m/2$ and $\mu = m$. First, it is immediate from the definition of ρ_n that

$$\exp(\rho_n^\mu) \sim \exp(n_0^{\mu/m}), \quad 1 \le \mu \le \left\lfloor \frac{m-1}{2} \right\rfloor. \tag{20}$$

So it remains to deal with the case $\mu = m$ for m odd and the cases $\mu = m/2, m$ for even m. For m even and $\mu = m$, the most complex subcase, we find that as $n \to \infty$

$$\begin{aligned}\rho_n^m &= n_0 (1+\Sigma)^{-m} \\ &= n_0 - c_m^{-1} n\Sigma + \frac{m+1}{2c_m} n\Sigma^2 + o(1) \\ &= n_0 - c_m^{-1} n\Sigma + \frac{m+1}{2c_m}\sigma^2 + o(1),\end{aligned}$$

and, hence

$$\exp(\rho_n^m) \sim \exp\left(n_0 - c_m^{-1} n\Sigma + \frac{m+1}{2c_m}\sigma^2\right) \quad (n \to \infty, 2 \mid m). \tag{21}$$

In a similar way we find that

$$\exp(\rho_n^{m/2}) \sim \exp\left(n_0^{1/2} - \frac{m}{2\sqrt{mc_m}}\sigma\right) \quad (n \to \infty, 2 \mid m) \tag{22}$$

and

$$\exp(\rho_n^m) \sim \exp\Big(n_0 - c_m^{-1} n\Sigma\Big) \quad (n \to \infty, 2 \nmid m). \tag{23}$$

Taking into account formulae (16), (18), (19), (20), (21), (22), and (23) we obtain (9), and Theorem 1 is proved. We record an interesting consequence of Theorem 1.

Corollary 1. *Let $P(z)$ and $Q(z)$ be polynomials meeting the requirements of Theorem 1. Put $\exp(P(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$ and $\exp(Q(z)) = \sum_{n=0}^{\infty} \beta_n z^n$, and assume that $\alpha_n \sim \beta_n$ as $n \to \infty$. Then we have $\alpha_n = \beta_n$ for all $n \geq 0$.*

It would be of interest to know whether this result remains true without the gap condition ($\mathscr{P}3$).

3. Asymptotic expansion of $e^{P(z)}$: The method of Harris and Schoenfeld

3.1. The result

In this paragraph we will provide an analogue of Stirling's asymptotic expansion (7) for a fairly large class of real polynomials $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu$ of degree $m > 1$. We also obtain a (rather complicated) analogue of the identity (8). In order to state our result we first give some technical definitions. In what follows all variables are integral with the usual convention that an empty sum equals zero. Define a sequence $\gamma_1, \gamma_2, \ldots$ by

$$\gamma_\mu = \beta_1^{-(\mu+1)} \sum_{\substack{\alpha \geq 0, \nu \geq 1 \\ \alpha m + \nu = \mu + 1}} \binom{-\nu/m}{\alpha} \frac{\beta_\nu}{\nu}, \tag{24}$$

where the β_ν's are given by (12). Next, we introduce a sequence $\mathscr{A}_1, \mathscr{A}_2, \ldots$ via the following hierarchy of definitions:

(i) $$\psi_{\alpha,\lambda} := (m^2 c_m)^{-\alpha} \sum_{\substack{\mu_1 + \ldots + \mu_\alpha = \alpha m - \lambda \\ 1 \leq \mu_1, \ldots, \mu_\alpha \leq m/2}} \mu_1^2 \ldots \mu_\alpha^2 c_{\mu_1} \ldots c_{\mu_\alpha},$$

(ii) $\chi_{\mu,k,\kappa} :=$

$$\left(\frac{m^2 c_m}{2}\right)^{-(k+\kappa)} \sum_{\substack{j_1 + \ldots + j_\kappa = 2k \\ j_1, \ldots, j_\kappa \geq 1}} \sum_{\substack{\mu_1 + \ldots + \mu_\kappa = \mu \\ 1 \leq \mu_1, \ldots, \mu_\kappa \leq m}} \prod_{\ell=1}^{\kappa} \left[\frac{\mu_\ell c_{\mu_\ell}}{j_\ell + 2} \left((-1)^{j_\ell} + \binom{\mu_\ell - 1}{j_\ell + 1}\right)\right],$$

(iii) $\widetilde{\chi}_{\nu,k,\kappa} :=$

$$\chi_{m(k+\kappa)-\nu,k,\kappa} + \sum_{\substack{\lambda, \mu \geq 1 \\ m(k+\kappa)+\lambda = \mu + \nu}} \sum_{\alpha \geq 1} (-1)^\alpha \binom{k + \kappa + \alpha - 1}{\alpha} \psi_{\alpha,\lambda} \chi_{\mu,k,\kappa},$$

(iv) $$\chi_\nu := \sum_{k \geq 1} \sum_{\kappa=1}^{2k} (-1)^{k+\kappa} 2^{1-2(k+\kappa)} \frac{(2k + 2\kappa - 1)!}{\kappa!(k + \kappa - 1)!} \widetilde{\chi}_{\nu,k,\kappa},$$

(v) $\widetilde{\chi}_\nu := \chi_\nu + \sum_{\beta\geq 1}\sum_{\lambda\geq 1}\binom{\beta}{\lambda}\chi_\beta \sum_{\substack{\nu_1+\ldots+\nu_\lambda=\nu-\beta\\ \nu_1,\ldots,\nu_\lambda\geq m/2}} \gamma_{\nu_1}\cdots\gamma_{\nu_\lambda}$,

(vi) $\mathscr{A}_\nu := \sum_{\lambda\geq 1}(-1)^{\lambda-1}\frac{1}{\lambda}\sum_{\substack{\nu_1+\ldots+\nu_\lambda=\nu\\ \nu_1,\ldots,\nu_\lambda\geq 1}}\widetilde{\chi}_{\nu_1}\cdots\widetilde{\chi}_{\nu_\lambda}$.

Finally, we define a sequence $\mathscr{B}_1, \mathscr{B}_2, \ldots$ by

$$
\begin{aligned}
\mathscr{B}_\nu &= \mathscr{B}_\nu(P) := \\
&\mathscr{A}_\nu + mc_m\sum_{\lambda\geq 1}(-1)^\lambda\frac{1}{\lambda}\left(\frac{\psi_{\lambda,\nu}}{2mc_m} - \sum_{\substack{\nu_1+\ldots+\nu_\lambda=m+\nu\\ \nu_1,\ldots,\nu_\lambda\geq m/2}}\gamma_{\nu_1}\cdots\gamma_{\nu_\lambda}\right)\\
&+\sum_{1\leq\mu\leq m}\sum_{\lambda\geq 1}(-1)^\lambda\binom{\mu+\lambda-1}{\lambda}c_\mu\sum_{\substack{\nu_1+\ldots+\nu_\lambda=\mu+\nu\\ \nu_1,\ldots,\nu_\lambda\geq m/2}}\gamma_{\nu_1}\cdots\gamma_{\nu_\lambda}\\
&+\sum_{\mu\geq 0}\sum_{\alpha\geq 1}\sum_{\lambda\geq 1}(-1)^{\alpha+\lambda}\frac{1}{2\lambda}(m^2c_m)^{-\lambda}\\
&\qquad\sum_{\substack{\nu_1+\ldots+\nu_\alpha=\nu-\mu\\ \nu_1,\ldots,\nu_\alpha\geq m/2}}\gamma_{\nu_1}\cdots\gamma_{\nu_\alpha}\sum\prod_{\ell=1}^{\lambda}\left[\mu_\ell^2c_{\mu_\ell}\binom{\mu_\ell+\alpha_\ell-1}{\alpha_\ell}\right],
\end{aligned}
\tag{25}
$$

the last sum being extended over those (2λ)-tuples $(\mu_1,\ldots,\mu_\lambda,\alpha_1,\ldots,\alpha_\lambda)$ for which $\mu_1+\ldots+\mu_\lambda = \lambda m-\mu$, $\alpha_1+\ldots+\alpha_\lambda=\alpha$, $1\leq\mu_1,\ldots,\mu_\lambda\leq m$, $\alpha_1,\ldots,\alpha_\lambda\geq 0$, and $(\mu_1,\alpha_1),\ldots,(\mu_\lambda,\alpha_\lambda)\neq(m,0)$.

Having given these definitions we can now state our result.

Theorem 2. *Suppose that the polynomial $P(z)=\sum_{\mu=1}^{m}c_\mu z^\mu\in\mathbb{R}[z]$ has degree $m\geq 2$ and meets the conditions ($\mathscr{P}1$), ($\mathscr{P}2$), and ($\mathscr{P}3$). Then there exist constants $\mathscr{C}_\nu = \mathscr{C}_\nu(P)$ such that the coefficients α_n of the entire function $\exp(P(z)) = \sum_{n=0}^{\infty}\alpha_n z^n$ have the asymptotic expansion*

$$
\alpha_n \approx \frac{K}{\sqrt{2\pi n}}n_0^{-n/m}\exp\left(P\left(n_0^{1/m}\right)\right)\left\{1+\sum_{\nu=1}^{\infty}\mathscr{C}_\nu n_0^{-\nu/m}\right\}\quad (n\to\infty),
\tag{26}
$$

where $K=K(P)$ and n_0 are as in Theorem 1. Moreover, the $\mathscr{C}_\nu$'s satisfy the (formal) identity

$$
1+\sum_{\nu=1}^{\infty}\mathscr{C}_\nu z^{-\nu} = \exp\left(\sum_{\nu=1}^{\infty}\mathscr{B}_\nu z^{-\nu}\right),
\tag{27}
$$

where the $\mathscr{B}_\nu = \mathscr{B}_\nu(P)$ are given by (25).

Remark. For $\nu \leq 3$ the coefficients $\mathscr{C}_\nu$ in (26) are given by (27) as

$$\mathscr{C}_1 = \begin{cases} \frac{c_1}{8c_2}\left(1+\frac{c_1^2}{12c_2}\right); & m=2 \\ -\frac{1}{2}\left(\frac{m-1}{2m}\right)^2 \frac{c_{(m-1)/2}^2}{c_m}; & 2\nmid m \\ -\frac{m-2}{4m}\frac{c_{m/2}c_{m/2-1}}{c_m}; & 2\mid m \text{ and } m>2, \end{cases}$$

$$\mathscr{C}_2 = \begin{cases} \frac{1}{4c_2}\left(\frac{1}{9}\left(\frac{c_1^2}{8c_2}\right)^3 + \frac{1}{3}\left(\frac{c_1^2}{8c_2}\right)^2 - \frac{c_1^2}{32c_2} - \frac{1}{3}\right); & m=2 \\ \frac{c_1}{9c_3}\left(1+\frac{c_1^3}{72c_3}\right); & m=3 \\ \frac{1}{8c_4}\left(\frac{c_2^3}{12c_4} + \frac{c_1^2c_2^2}{16c_4} - \frac{c_1^2}{4} + c_2\right); & m=4 \\ \frac{m-1}{4m^2}\frac{c_{(m-1)/2}}{c_m}\left(\frac{(m-1)^3}{32m^2}\frac{c_{(m-1)/2}^3}{c_m} - (m-3)c_{(m-3)/2}\right); & 2\nmid m \text{ and } m>3 \\ \frac{1}{2c_m}\left(\left(\frac{m-2}{4m}\right)^2\frac{c_{m/2}^2c_{m/2-1}^2}{c_m} - \left(\frac{m-2}{2m}\right)^2 c_{m/2-1}^2 \right. & \\ \qquad\qquad \left. -\frac{m-4}{2m}c_{m/2}c_{m/2-2}\right); & 2\mid m \text{ and } m>4, \end{cases}$$

and

$$\mathscr{C}_3 = \begin{cases} \frac{c_1}{384c_2^2}\left(\frac{1}{54}\left(\frac{c_1^2}{4c_2}\right)^4 + \frac{1}{6}\left(\frac{c_1^2}{4c_2}\right)^3 - \frac{11}{10}\left(\frac{c_1^2}{4c_2}\right)^2 - \frac{35c_1^2}{24c_2} - 1\right); & m=2 \\ -\frac{1}{6c_3}\left(\frac{1}{2}\left(\frac{c_1^3}{54c_3}\right)^2 + \frac{c_1^3}{27c_3} + \frac{1}{2}\right); & m=3 \\ \frac{c_1}{32c_4}\left(\frac{c_1^2c_2}{8c_4} - \frac{c_1^2c_2^3}{96c_4^2} - \frac{c_2^4}{24c_4^2} - \frac{5c_2^2}{4c_4} + \frac{c_1^2}{c_4} + 3\right); & m=4 \\ \frac{1}{25c_5}\left(\frac{4c_1c_2^3}{25c_5} - \frac{4c_2^6}{1875c_5^2} - \frac{c_1^2}{2} + 3c_2\right); & m=5 \\ \frac{1}{2c_6}\left(\frac{c_1c_2c_3^2}{36c_6} - \frac{c_2^3c_3^3}{648c_6^2} + \frac{c_2^3c_3}{54c_6} + \frac{c_3^3}{48c_6} - \frac{c_1c_2}{9} + \frac{c_3}{4}\right); & m=6 \\ \frac{1}{4m^2c_m}\left(\frac{(m-1)^3(m-3)}{8m^2}\frac{c_{(m-1)/2}^3c_{(m-3)/2}}{c_m} \right. & \\ \qquad -\frac{1}{3}\left(\frac{(m-1)^3}{16m^2}\right)^2\left(\frac{c_{(m-1)/2}^3}{c_m}\right)^2 - \frac{(m-3)^2}{2}c_{(m-3)/2}^2 & \\ \qquad\qquad \left. -(m-1)(m-5)c_{(m-1)/2}c_{(m-5)/2}\right); & 2\nmid m \text{ and } m>5 \\ \frac{1}{4mc_m}\left(\frac{(m-2)^3}{8m^2}\frac{c_{m/2}c_{m/2-1}^3}{c_m} - \frac{(m-2)^3}{96m^2}\frac{c_{m/2}^3c_{m/2-1}^3}{c_m^2} \right. & \\ \qquad\qquad +\frac{(m-2)(m-4)}{4m}\frac{c_{m/2}^2c_{m/2-1}c_{m/2-2}}{c_m} & \\ \quad \left. -\frac{(m-2)(m-4)}{m}c_{m/2-1}c_{m/2-2} - (m-6)c_{m/2}c_{m/2-3}\right); & 2\mid m \text{ and } m>6. \end{cases}$$

More generally, the calculation of $\mathscr{C}_\nu$ for arbitrary ν will require the distinction of the $2\nu+1$ cases $m=2,3,\ldots,2\nu$, $2\nmid m$ and $m>2\nu-1$, $2\mid m$ and $m>2\nu$.

The proof of Theorem 2 proceeds in principle along similar lines as the proof of Theorem 1; there are however additional complications both of a theoretical and a technical sort of nature. The main problem is to establish an appropriate refinement of formula (10), again in terms of the positive real root of a polynomial equation. This is accomplished in Subsections 3 - 5 building on a result of Harris and Schoenfeld described in the next subsection. The rest of the proof of Theorem 2 is then sketched in Subsection 6.

3.2. The method of Harris and Schoenfeld

Nowadays there exists a range of methods for dealing with the problem of obtaining asymptotic information on the coefficients α_n of a function $f(z)=\sum\limits_{n=0}^{\infty} \alpha_n z^n$ analytic in some neighborhood of the origin. Here we mention only the paper [6] of Hayman cited in the previous paragraph and the work of Harris and Schoenfeld; cf. [7] and [8]. Hayman derives an asymptotic formula for α_n under relatively mild conditions on $f(z)$. In their paper [8] Harris and Schoenfeld study analytic functions satisfying considerably more stringent regularity conditions and obtain for the coefficients of these functions complete asymptotic expansions. For the convenience of the reader we briefly explain here a special case of their result, which will be used in the proof of Theorem 2.

Call a function $f(z)$ *HS-admissible in* $|z| < R$ provided $f(z)$ satisfies the following conditions of Harris and Schoenfeld.

(A) $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ is analytic for $|z|<R$, $0<R\leq\infty$, and is real for real z.

(B) There exists $R_0\in(0,R)$ and a real-valued function $d(r)$ defined on (R_0,R) such that, for any $r\in(R_0,R)$, $0<d(r)<1$ and $r(1+d(r))<R$. Moreover, $f(z)\neq 0$ for each z such that $|z-r|\leq rd(r)$ and every $r\in(R_0,R)$.

(C) Defining, for $k\geq 1$,

$$A(z)=\frac{f'(z)}{f(z)},\quad B_k(z)=\frac{z^k}{k!}A^{(k-1)}(z),\quad B(z)=\frac{1}{2}zB_1'(z),$$

we have $B(r)>0$ for $R_0<r<R$ and $B_1(r)\to\infty$ as $r\to R^-$.

(D) For suitable R_1 and large n define r_n to be the unique solution of the equation $B_1(r)=n+1$ which satisfies $R_1<r<R$. For $j=1,2,\ldots$ define

$$C_j(r)=-\left\{B_{j+2}(r)+\frac{(-1)^j}{j+2}B_1(r)\right\}\Big/B(r),$$

and suppose that there exist non-negative D_n, E_n, and a positive integer n_1 such that the inequality

$$|C_j(r_n)|\leq E_n D_n^j$$

holds for all $n\geq n_1$ and all $j\geq 1$.

(E) As $n\to\infty$, we have

$$B(r_n)\,(d(r_n))^2 \to \infty, \quad D_nE_nB(r_n)\,(d(r_n))^3 \to 0, \quad D_nd(r_n) \to 0.$$

$f(z)$ is called *HS-admissible* if it is HS-admissible in $|z| < R$ for some R with $0<R\leq\infty$. Harris and Schoenfeld prove the following result.

Theorem ([8, Theorem 1]). *Suppose that* $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ *is HS-admissible. Then we have for every integer* $N\geq 0$ *that*

$$\alpha_n = \frac{f(r_n)}{2r_n^n\sqrt{\pi B(r_n)}}\left\{1+\sum_{k=1}^{N}\frac{F_k(n)}{(B(r_n))^k}+\mathcal{O}\left(\varphi_N(n;d)\right)\right\} \quad (n\to\infty), \tag{28}$$

where

$$F_k(n) := \frac{(-1)^k}{\sqrt{\pi}}\sum_{\mu=1}^{2k}\frac{\Gamma(\mu+k+1/2)}{\mu!}\sum_{\substack{j_1+\dots+j_\mu=2k\\ j_1,\dots,j_\mu\geq 1}} C_{j_1}(r_n)\dots C_{j_\mu}(r_n),$$

$\lambda(r;d)$ is the maximum value of $|f(z)/f(r)|$ for z on the oriented path $Q=Q(r;d)$ consisting of the line segment $L=L(r;d)$ from $r+ird(r)$ to $r\sqrt{1-(d(r))^2}+ird(r)$ and of the circular arc $C=C(r;d)$ from the last point to ir to $-r$,

$$\mu(r;d) := \max\left(\lambda(r;d)\sqrt{B(r)},\ \frac{\exp(-B(r)(d(r))^2)}{d(r)\sqrt{B(r)}}\right),$$

$E_n' := \min(1,E_n)$, $\quad E_n'' := \max(1,E_n)$, *and*

$$\varphi_N(n;d) := \max\left(\mu(r_n;d),\, E_n'\left(D_nE_n''/\sqrt{B(r_n)}\right)^{2N+2}\right).$$

3.3. HS-admissibility of $\exp(P(z))$

Given a polynomial $P(z)\in\mathbb{C}[z]$ we provide a criterion for the function $f(z)=\exp(P(z))$ to be HS-admissible.

Lemma 1. *Let* $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu$ *be a polynomial of exact degree* $m \geq 1$ *with complex coefficients* c_μ, *and let* $f(z)=\exp(P(z))$. *Then the following assertions are equivalent:*

(i) $f(z)$ *is HS-admissible.*

(ii) $f(z)$ *is HS-admissible in the whole complex plane.*

(iii) $P(z)\in\mathbb{R}[z]$ *and* $c_m>0$.

Proof. $f(z)$ is an entire function and we have

$$A(z) = P'(z), \quad B_k(z) = \frac{z^k}{k!}P^{(k)}(z), \quad B(z) = \frac{1}{2}z\left(P'(z)+zP''(z)\right).$$

Since $B_1(r)=\sum_{\mu=1}^{m} \mu c_\mu r^\mu$ remains bounded on any finite interval the condition that $B_1(r)\to\infty$ $(r\to R^-)$ implies $R=\infty$, whence the equivalence of (i) and (ii). We show that (ii)$\Longleftrightarrow$(iii). Suppose first that $P(z)\in\mathbb{R}[z]$ and that c_m is positive. Then we have

$$B(r) = \sum_{\mu=1}^{m} \frac{\mu^2}{2}c_\mu r^\mu = \frac{m^2}{2}c_m r^m\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right)>0$$

for all sufficiently large r, $r>R_0>0$ say, and

$$B_1(r) = \sum_{\mu=1}^{m} \mu c_\mu r^\mu = m c_m r^m\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right)\to\infty$$

as $r\to\infty$. Also, $f(z)$ is real for real z. Hence, conditions (A) - (C) hold for $R=\infty$ and any function $d(r)$ on (R_0,∞) satisfying $0<d(r)<1$ with R_0 as chosen above.

By our assumption (iii), r_n is well-defined and strictly increasing for all sufficiently large n and $r_n\to\infty$ as $n\to\infty$, and hence $r_n\sim n_0^{1/m}$. We have

$$C_j(r) = -\frac{W_j(r)}{B(r)},$$

where

$$W_j(r) := \frac{1}{j+2}\sum_{\mu=1}^{m}\left((-1)^j+\binom{\mu-1}{j+1}\right)\mu c_\mu r^\mu \quad (j=1,2,\ldots)$$

is a polynomial in r of degree at most m. Now for $r\geq1$ and each $j=1,2,\ldots$

$$|W_j(r)| = \left|\sum_{\mu=1}^{m}(j+2)^{-1}\left((-1)^j+\binom{\mu-1}{j+1}\right)\mu c_\mu r^\mu\right|$$

$$\leq \sum_{\mu=1}^{m} (j+2)^{-1}\left(1+\binom{\mu-1}{j+1}\right)\mu|c_\mu|r^\mu$$
$$\leq \sum_{\mu=1}^{m} \frac{1}{3}\left(1+\binom{\mu-1}{\lfloor\frac{\mu-1}{2}\rfloor}\right)\mu|c_\mu|r^\mu$$
$$\leq \frac{1}{3}\left(1+\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}\right) mc_m r^m(1+Cr^{-1}),$$

where $C\geq 0$ is independent of r and j; in fact one can take

$$C := (m-1)c_m^{-1}\max_{1\leq\mu\leq m}|c_\mu|.$$

Also, for $r\geq\max(1,C)$,

$$|B(r)| = \left|\sum_{\mu=1}^{m}\frac{\mu^2}{2}c_\mu r^\mu\right| \geq \left|\frac{m^2}{2}c_m r^m - \left|\sum_{\mu<m}\frac{\mu^2}{2}c_\mu r^\mu\right|\right|$$
$$\geq \frac{m^2}{2}c_m r^m(1-Cr^{-1})$$

with C as above. It follows that for all sufficiently large r and each $j\geq 1$

$$|C_j(r)| \leq \frac{2}{3}m^{-1}\left(1+\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}\right)(1+Cr^{-1})(1-Cr^{-1})^{-1}$$

and that there exists some number r_0 such that for $r\geq r_0$ and each $j=1,2,\ldots$

$$|C_j(r)| \leq m^{-1}\left(1+\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}\right) =: m_0.$$

In particular, since $r_n\to\infty$ and is strictly increasing for large n, there exists a positive integer n_1 such that the estimate $|C_j(r_n)|\leq m_0$ holds for all $n\geq n_1$ and every $j\geq 1$. Hence, we can take $E_n = m_0$ and $D_n = 1$, in this way satisfying condition (D). Also,

$$B(r_n) = \sum_{\mu=1}^{m}\frac{\mu^2}{2}c_\mu r_n^\mu \sim \frac{mn}{2}.$$

Putting $\delta_n := d(r_n)$ we will have all conditions (A) - (E) satisfied with $R=\infty$, provided we can determine $d(r)$ so that $0<d(r)<1$ and

$$n\delta_n^2\to\infty,\quad n\delta_n^3\to 0\quad (n\to\infty). \tag{29}$$

It is, of course, trivial to find functions $d(r)$ satisfying (29) and $0<d(r)<1$ for sufficiently large r. For example, a function $d(r)=r^{-\alpha}$, $\alpha\in\mathbb{R}$, will satisfy (29) whenever $m/3<\alpha<m/2$. Hence, after readjusting R_0 if necessary, the implication (iii) $\Longrightarrow$ (ii) is proved.

Suppose, conversely, that $f(z)$ is HS-admissible in the whole complex plane. By (A), $f(r)=\exp(P(r))$ has to be real for each $r\in\mathbb{R}$. But this implies that

$$\Omega := \left\{ \sum_{\mu=1}^{m} \operatorname{Im}(c_\mu) r^\mu : r \in \mathbb{R} \right\} \subseteq \mathbb{Z}\pi,$$

in particular Ω is a discrete set, which is impossible unless $c_\mu \in \mathbb{R}$ for all $\mu = 1,2,\dots,m$. For, if $\operatorname{Im}(c_\mu) \neq 0$ for some μ then Ω contains a half-line. Hence, we conclude that $P(z)\in\mathbb{R}[z]$. Moreover, it follows that $c_m>0$, for otherwise we would have

$$B_1(r) = \sum_{\mu=1}^{m} \mu c_\mu r^\mu \to -\infty \quad (r \to \infty),$$

contradicting (C). ∎

Remark. As a rule, the assumptions (A) - (E) of Harris and Schoenfeld on a function $f(z)$ are harder to verify than the corresponding conditions of Hayman, which we shall refer to as H-admissibility. It is not true however, that HS-admissibility implies H-admissibility. In fact, Lemma 1 just proved in conjunction with Hayman's theorem [6, Theorem X] tells us that, on the contrary, for the class of functions $\mathcal{F} = \{e^{P(z)} : P(z) \in \mathbb{R}[z]\}$ H-admissibility implies HS-admissibility and that there exist HS-admissible functions in $\mathcal{F}$ which are not H-admissible; for example $e^{z(z-1)}$ has this property. However, a marked difference between these two methods is that while Hayman's method immediately supplies a meaningful asymptotic result for the coefficients of an analytic function whose H-admissibility has been established, application of the Harris-Schoenfeld method in addition to verifying its assumptions also depends on the estimation of the error term $\varphi_N(n;d)$ for the given function $f(z)$. If, on the other hand, we impose on the polynomial $P(z)$ the conditions of Proposition 1 below, which aside from the HS-admissibility also guarantee that $\varphi_N(n;d)$ can be bounded appropriately, we obtain a class of H-admissible functions in $\mathcal{F}$.

3.4. Estimation of $\varphi_N(n;d)$

Lemma 2. *Suppose that* $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu \in \mathbb{R}[z]$ *has degree* $m \geq 2$ *and meets the conditions* $(\mathcal{P}1)$ *and* $(\mathcal{P}2)$. *Then the function* $f(z) = \exp(P(z))$ *is HS-admissible and the auxiliary function* $d(r)$ *can be chosen in such a way that for each fixed*

$N \geq 0$

$$\varphi_N(n;d) = \mathcal{O}(1/n^{N+1}) \quad (n \to \infty). \tag{30}$$

Proof. The HS-admissibility of $f(z)$ follows from Lemma 1. Moreover, the proof of Lemma 1 shows that in order to satisfy the assumptions (A) - (E) for the function $f(z)$ we can take $D_n := 1$, $E_n := m_0 = m^{-1}\left(1+\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}\right)$, and $d(r) = r^{-\alpha}$ with $m/3 < \alpha < m/2$. With this choice of D_n and E_n we have $E_n' = 1$, $E_n'' = E_n = m_0$, and, for fixed N,

$$E_n' \left(\frac{D_n E_n''}{\sqrt{B(r_n)}}\right)^{2N+2} \sim (2m_0^2/m)^{N+1} n^{-(N+1)} \quad (n \to \infty);$$

in particular the left-hand side is precisely of order $\mathcal{O}(1/n^{N+1})$. We will select $d(r)$ so that $\mu(r_n;d)$ is of smaller order of magnitude than the above term, which will then provide the estimate (30) on the error term $\varphi_N(n;d)$ in (28).

We have $|f(z)| = \exp(\mathrm{Re}(P(z)))$ and on setting $z = x+iy = |z|e^{i\vartheta}$,

$$\mathrm{Re}\,(P(z)) = \sum_{\mu=1}^{m} \sum_{0\leq\nu\leq\mu/2} (-1)^\nu \binom{\mu}{2\nu} c_\mu x^{\mu-2\nu} y^{2\nu} = \sum_{\mu=1}^{m} c_\mu |z|^\mu \cos(\mu\vartheta).$$

If z is on $L = L(r;d)$ then we have $0 < r\sqrt{1-(d(r))^2} \leq x \leq r$ and $y = rd(r)$; hence, using the facts that $m \geq 2$ and that the coefficients of $P(z)$ are non-negative we find by separating summands according to $\nu = 0$, $\nu = 1$, or $\nu \geq 2$ that for large enough r

$$\begin{aligned}
\mathrm{Re}\,(P(z)) &= \sum_{\mu=1}^{m} \sum_{0\leq\nu\leq\mu/2} (-1)^\nu \binom{\mu}{2\nu} c_\mu x^{\mu-2\nu} y^{2\nu} \\
&\leq P(r) - \binom{m}{2} c_m \left(1-(d(r))^2\right)^{\frac{m-2}{2}} r^m (d(r))^2 + o\left(r^m (d(r))^2\right) \\
&\leq P(r) - \left(\frac{m-1}{2}\right)^2 c_m r^m (d(r))^2,
\end{aligned}$$

provided that $d(r) \to 0$ as $r \to \infty$. It follows that under this assumption on $d(r)$ for $z \in L$ and sufficiently large r

$$|f(z)/f(r)| \leq \exp\left(-\left(\frac{m-1}{2}\right)^2 c_m r^m (d(r))^2\right).$$

If z is on $C = C(r;d)$ then

$$\mathrm{Re}\,(P(z)) = \sum_{\mu=1}^{m} c_\mu r^\mu \cos(\mu\vartheta),$$

and we have $0<\vartheta_0\leq\vartheta\leq\pi$, where $\sin(\vartheta_0)=d(r)$. Clearly,

$$\cos(\vartheta)\leq\cos(\pi/m)<1,\quad \pi/m\leq\vartheta\leq\pi$$

and

$$\cos(m\vartheta)\leq\cos(m\vartheta_0)<1,\quad \vartheta_0\leq\vartheta\leq\pi/m.$$

Using ($\mathcal{P}1$) and ($\mathcal{P}2$) this yields the estimate

$$\operatorname{Re}(P(z))\leq\begin{cases}P(r)-Ar; & \pi/m\leq\vartheta\leq\pi\\ P(r)-c_m(1-\cos(m\vartheta_0))\,r^m; & \vartheta_0\leq\vartheta\leq\pi/m,\end{cases}$$

where $A:=c_1(1-\cos(\pi/m))>0$. Assuming again that $d(r)\to 0$ $(r\to\infty)$ we have that as $r\to\infty$

$$\vartheta_0=d(r)+\mathcal{O}\left((d(r))^3\right),$$

$$\cos(m\vartheta_0)=1-\frac{m^2}{2}(d(r))^2+\mathcal{O}\left((d(r))^4\right),$$

and hence, for sufficiently large r, say,

$$\cos(m\vartheta_0)\leq 1-\frac{m^2}{4}(d(r))^2.$$

We conclude that for $z\in C$ and large r

$$\operatorname{Re}(P(z))\leq P(r)-\min\left\{Ar,\frac{m^2}{4}c_m r^m(d(r))^2\right\},$$

and therefore in this case

$$|f(z)/f(r)|\leq\max\left\{\exp(-Ar),\exp\left(-\frac{m^2}{4}c_m r^m(d(r))^2\right)\right\}.$$

Summarizing, we find that there exists a number r_0 such that for $r\geq r_0$

$$\lambda(r;d)\leq\max\left\{\exp\left(-\left(\frac{m-1}{2}\right)^2 c_m r^m(d(r))^2\right),\exp(-Ar)\right\},$$

provided that $d(r)\to 0$ as $r\to\infty$. It follows that under these assumptions

$$\begin{aligned}\mu(r;d)=\mathcal{O}\Big(&\max\Big\{r^{m/2}\exp\Big(-\Big(\frac{m-1}{2}\Big)^2 c_m r^m(d(r))^2\Big),r^{m/2}\exp(-Ar),\\&\frac{r^{-m/2}}{d(r)}\exp\Big(-\frac{m^2}{2}c_m r^m(d(r))^2\Big)\Big\}\Big)\\=\mathcal{O}\Big(&r^{m/2}\max\Big\{\frac{1}{d(r)}\exp\Big(-\Big(\frac{m-1}{2}\Big)^2 c_m r^m(d(r))^2\Big),\exp(-Ar)\Big\}\Big).\end{aligned}$$

Now choose $d_0(r) := r^{-2m/5}$. Then $d_0(r) \to 0$ as $r \to \infty$ and conditions (29) on $\delta_n = d_0(r_n)$ hold. Moreover, for sufficiently large r

$$\mu(r; d_0) = \mathcal{O}\Big(\max\Big\{ \exp\Big(\frac{9}{10} m \log r - \Big(\frac{m-1}{2}\Big)^2 c_m r^{m/5}\Big), \exp\Big(\frac{m}{2}\log r - Ar\Big)\Big\}\Big).$$

Since $r_n \sim n_0^{1/m}$ we find that as $n \to \infty$

$$\begin{aligned}\mu(r_n; d_0) &= \mathcal{O}\Big(\max\Big\{ \exp\Big(\frac{9}{10}\log n - A_1 n^{1/5}\Big), \exp\Big(\frac{1}{2}\log n - A_2 n^{1/m}\Big)\Big\}\Big)\\ &= \mathcal{O}\left(\exp(-(N+2)\log n)\right) = \mathcal{O}\left(1/n^{N+2}\right)\end{aligned}$$

with constants $A_1, A_2 > 0$ which need not be specified and any fixed $N \geq 0$. It follows now indeed that

$$\varphi_N(n; d_0) = \mathcal{O}\left(1/n^{N+1}\right) \quad (n \to \infty),$$

and Lemma 2 is proved. ∎

3.5. An asymptotic expansion for α_n

We summarize the discussion of Subsections 2 – 4 as follows.

Proposition 1. *Suppose that* $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu \in \mathbb{R}[z]$ *has degree* $m \geq 2$ *and meets the conditions* ($\mathcal{P}1$) *and* ($\mathcal{P}2$). *Then the coefficients* α_n *of the entire function* $\exp(P(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$ *satisfy, for each fixed integer* $N \geq 0$, *the relation*

$$\alpha_n = \frac{\exp(P(r_n))}{2 r_n^n \sqrt{\pi B(r_n)}} \left\{ 1 + \sum_{k=1}^{N} \frac{P_k(r_n)}{(B(r_n))^{3k}} + \mathcal{O}\left(1/n^{N+1}\right)\right\} \quad (n \to \infty), \tag{31}$$

where

$$B(r) = \sum_{\mu=1}^{m} \frac{\mu^2}{2} c_\mu r^\mu,$$

$$P_k(r) = \frac{(-1)^k}{2^{6k-1}} \sum_{\mu=1}^{2k} (-1)^\mu 2^{2(2k-\mu)} \frac{(2\mu+2k-1)!}{\mu!(\mu+k-1)!} (B(r))^{2k-\mu} \cdot \sum_{\substack{j_1+\dots+j_\mu=2k\\ j_1,\dots,j_\mu \geq 1}} W_{j_1}(r) \dots W_{j_\mu}(r) \qquad (k = 1, 2, \dots),$$

$$W_j(r) = \sum_{\mu=1}^{m} (j+2)^{-1} \left((-1)^j + \binom{\mu-1}{j+1} \right) \mu c_\mu r^\mu \quad (j = 1, 2, \dots),$$

and r_n *is the positive real root of the equation* $\sum_{\mu=1}^{m} \mu c_\mu r^\mu = n+1$.

3.6. Proof of Theorem 2 (concluding remarks)

We now replace the root r_n of the equation $rP'(r) = n+1$ by a sequence of functions $\rho_n(s)$ giving better and better approximations for r_n^n and $\exp(P(r_n))$ as the parameter s increases. More specifically, in order to ensure that a function ρ_n satisfies $r_n^n = \rho_n^n \left(1+\mathcal{O}(n^{-s/m})\right)$ and $\exp(P(r_n)) = \exp(P(\rho_n)) \left(1+\mathcal{O}(n^{-s/m})\right)$ for some given integer $s \geq 1$, ρ_n has to approximate r_n with an (absolute) error of order $\mathcal{O}\left(n^{-1-(s-1)/m}\right)$. Proceeding by Lagrange inversion as described in the second paragraph we find that

$$r_n \approx n_0^{1/m} \left\{ 1 + \sum_{\mu=1}^{\infty} \gamma_\mu n_0^{-\mu/m} \right\}^{-1} \quad (n \to \infty), \tag{32}$$

where the γ_μ's are given by (24). For $s=1,2,\ldots$ define a function $\rho_n(s)$ by

$$\rho_n(s) := n_0^{1/m} \left\{ 1 + \sum_{\mu=1}^{m+s-1} \gamma_\mu n_0^{-\mu/m} \right\}^{-1}.$$

Clearly, this is well-defined for sufficiently large n, and by (32) we have that

$$r_n = \rho_n(s) + \mathcal{O}\left(n^{-1-(s-1)/m}\right) \quad (n \to \infty)$$

as required. Proposition 1 together with these observations now yields, for any choice of integers $s \geq 1$, $N \geq 0$, the relation

$$\alpha_n = \frac{\exp(P(\rho_n(s)))}{2(\rho_n(s))^n \sqrt{\pi B(\rho_n(s))}} \left\{ 1 + \sum_{k=1}^{N} \frac{P_k(\rho_n(s))}{(B(\rho_n(s)))^{3k}} + \mathcal{O}(n^{-s/m}) + \mathcal{O}\left(1/n^{N+1}\right) \right\} \quad (n \to \infty). \tag{33}$$

In (33) put $N = \lceil s/m - 1 \rceil$, so that $1/n^{N+1} = \mathcal{O}(n^{-s/m})$. Our final task is to transform the resulting expansion into the form (26). It is at this point that our assumption ($\mathcal{P}3$) comes into play. By ($\mathcal{P}3$) we have in particular that $\gamma_\mu = 0$ for $0 < \mu < m/2$ and

$$\gamma_{m/2} = \frac{c_{m/2}}{2mc_m}, \quad 2 \mid m.$$

Using these facts and dealing with the terms in the above expansion for fixed $s \geq 1$ much in the same way as was done in the proof of Theorem 1 we find that

$$\alpha_n = \frac{K}{\sqrt{2\pi n}} n_0^{-n/m} \exp\left(P(n_0^{1/m})\right) \exp\left(\sum_{\nu=1}^{s-1} \mathcal{B}_\nu n_0^{-\nu/m} + \mathcal{O}\left(n^{-s/m}\right) \right) \quad (n \to \infty)$$

with K and n_0 as in Theorem 1, and $\mathscr{B}_\nu$ as given by (25). Since s was arbitrary (26) follows, and by setting $z=n_0^{1/m}$ and letting $s\to\infty$ we obtain (27).

4. Entire functions of finite genus with finitely many zeros

The problem to obtain asymptotic information on the coefficients of analytic functions belonging to the class described in the title was, as far as we are aware, first studied by Pólya [13] in connection with his investigation concerning functions having the property that all of their derivatives possess only real zeros. Entire functions of finite genus having only finitely many zeros are of the form

$$f(z) = Q(z)e^{P(z)} = \sum_{n=0}^{\infty} \beta_n z^n$$

with polynomials $P(z)$ and $Q(z)$, and Pólya's result [13, formula (53)] provides an asymptotic formula for β_n containing certain unspecified constants. Here, we apply Theorems 1 and 2 to obtain completely explicit results. In what follows $P(z)=\sum_{\mu=1}^{M} c_\mu z^\mu \in \mathbb{R}[z]$ and $Q(z)=\sum_{\nu=0}^{N} d_\nu z^\nu \in \mathbb{C}[z]$ are polynomials of degree $M\geq 1$ and $N\geq 0$, respectively. Theorem 3 below describes the asymptotic behaviour of the quotient α_n/α_{n-1} of successive coefficients for the function $\exp(P(z))=\sum_{n=0}^{\infty}\alpha_n z^n$ under the hypotheses of Theorems 1 or 2. From Theorems 1-3 the desired results on the coefficients β_n are then readily derived. Theorem 3 or, rather, the special case considered in the last paragraph, also plays a role in the investigation of the subgroup growth for virtually free groups; cf. [9].

Theorem 3. *Let* $P(z)$ *be as above and consider the entire function* $\exp(P(z)) = \sum_{n=0}^{\infty}\alpha_n z^n$.

(i) *If* $P(z)$ *meets the conditions* ($\mathscr{P}0$) *and* ($\mathscr{P}3$) *then we have*

$$\alpha_n/\alpha_{n-1} \sim n_0^{-1/M} \quad (n\to\infty). \tag{34}$$

(ii) *If* $M\geq 2$ *and* $P(z)$ *meets the conditions* ($\mathscr{P}1$), ($\mathscr{P}2$), *and* ($\mathscr{P}3$) *then there exist constants* $\widetilde{Q}_\nu=\widetilde{Q}_\nu(P)$ *such that* α_n/α_{n-1} *has the asymptotic expansion*

$$\alpha_n/\alpha_{n-1} \approx n_0^{-1/M}\left\{1+\sum_{\nu=1}^{\infty}\widetilde{Q}_\nu n_0^{-\nu/M}\right\} \quad (n\to\infty). \tag{35}$$

Moreover, the $\widetilde{Q}_\nu$ *satisfy the identity*

$$1+\sum_{\nu=1}^{\infty}\widetilde{Q}_\nu z^{-\nu} = \exp\left(\sum_{\nu=1}^{\infty} Q_\nu z^{-\nu}\right), \tag{36}$$

where

$$Q_\nu = Q_\nu(P) := -\sum_{1\leq\lambda<\nu/M}\binom{\nu/M-1}{\lambda}(Mc_M)^{-\lambda}\mathscr{B}_{\nu-M\lambda}(P)$$

$$+\begin{cases} -\frac{\nu+M-2}{2\nu(\nu+M)M^{\nu/M-1}c_M^{\nu/M}}; & \nu\equiv 0(M)\\ 0; & \nu\equiv i(M),\ 1\leq i<M/2\\ (-1)^{(\nu-i)/M}\binom{1-i/M}{1+(\nu-i)/M}(Mc_M)^{-1-(\nu-i)/M}c_{M-i}; & \nu\equiv i(M),\ M/2\leq i<M. \end{cases}$$

Proof. Since the right-hand side of (9) coincides with the main term in the expansion (26) it is enough to consider (ii). By Theorem 2 the quotient α_n/α_{n-1} has the asymptotic expansion

$$\alpha_n/\alpha_{n-1} \approx$$
$$n_0^{-1/M}\left(1-\frac{1}{n}\right)^{(n-1)/M+1/2}\exp\left(\sum_{\mu=1}^{M}c_\mu(Mc_M)^{-\mu/M}\left(n^{\mu/M}-(n-1)^{\mu/M}\right)\right)$$
$$\times\frac{1+\sum_{\nu=1}^{\infty}\mathscr{C}_\nu(Mc_M)^{\nu/M}n^{-\nu/M}}{1+\sum_{\nu=1}^{\infty}\mathscr{C}_\nu(Mc_M)^{\nu/M}(n-1)^{-\nu/M}}\quad(n\to\infty),$$

where the $\mathscr{C}_\nu$'s are given by (27). Now, for $n\geq 2$

$$\left(1-\frac{1}{n}\right)^{(n-1)/M+1/2} = \exp\left(\left(\frac{n-1}{M}+\frac{1}{2}\right)\log\left(1-\frac{1}{n}\right)\right)$$
$$= e^{-1/M}\exp\left(-\sum_{\lambda=1}^{\infty}\frac{M(\lambda+1)-2}{2\lambda(\lambda+1)M^{\lambda+1}c_M^{\lambda}}n_0^{-\lambda}\right)$$

and

$$\exp\left(\sum_{\mu=1}^{M}c_\mu(Mc_M)^{-\mu/M}\left(n^{\mu/M}-(n-1)^{\mu/M}\right)\right) =$$
$$\exp\left(\sum_{\mu=1}^{M}c_\mu(Mc_M)^{-\mu/M}\sum_{\lambda=1}^{\infty}(-1)^{\lambda-1}\binom{\mu/M}{\lambda}n^{\mu/M-\lambda}\right) =$$
$$e^{1/M}\exp\Bigg(\sum_{\substack{\nu\geq 1\\ \nu\equiv i(M)\\ M/2\leq i<M}}(-1)^{\lfloor\nu/M+1\rfloor-1}\binom{\lfloor\nu/M+1\rfloor-\nu/M}{\lfloor\nu/M+1\rfloor}.$$

$$(Mc_M)^{-\lfloor \nu/M+1\rfloor} c_{M\lfloor \nu/M+1\rfloor-\nu} n_0^{-\nu/M}\Bigg).$$

Also, by (27), we have for $n \geq 2$ that

$$\frac{1+\sum_{\nu=1}^{\infty} \mathscr{C}_\nu (Mc_M)^{\nu/M} n^{-\nu/M}}{1+\sum_{\nu=1}^{\infty} \mathscr{C}_\nu (Mc_M)^{\nu/M} (n-1)^{-\nu/M}} =$$

$$\exp\left(\sum_{\nu=1}^{\infty} \mathscr{B}_\nu (Mc_M)^{\nu/M}\left(n^{-\nu/M}-(n-1)^{-\nu/M}\right)\right) =$$

$$\exp\left(-\sum_{\nu=M+1}^{\infty}\left(\sum_{1\leq\lambda<\nu/M}\binom{\nu/M-1}{\lambda}(Mc_M)^{-\lambda}\mathscr{B}_{\nu-M\lambda}(P)\right) n_0^{-\nu/M}\right).$$

Assertion (ii) follows now since the convergent power series involved in these formulae are also asymptotic as $n \to \infty$. ∎

The coefficients β_n of $f(z) = Q(z)e^{P(z)}$ are related to the coefficients α_n of $e^{P(z)}$ by

$$\beta_n = d_N \alpha_{n-N}\left\{1+\sum_{0\leq\nu<N} d_\nu d_N^{-1} \alpha_{n-\nu}/\alpha_{n-N}\right\}, \quad n \geq N. \tag{37}$$

Assume that $P(z)$ satisfies the hypotheses of Theorem 2. By iteration we obtain from (35) and (36) that for fixed $k \geq 1$ as $n \to \infty$

$$\alpha_n/\alpha_{n-k} \approx n_0^{-k/M} \exp\left(\sum_{\gamma=1}^{\infty} Q_\gamma^{(k)} n_0^{-\gamma/M}\right), \tag{38}$$

where

$$Q_\gamma^{(k)} = Q_\gamma^{(k)}(P) := kQ_\gamma(P) + \sum_{1\leq\lambda<\gamma/M}\binom{\gamma/M-1}{\lambda}(Mc_M)^{-\lambda} Q_{\gamma-\lambda M}(P)\mathfrak{s}_\lambda(k-1) + \begin{cases} \frac{\mathfrak{s}_{\gamma/M}(k-1)}{(Mc_M)^{\gamma/M}\gamma}; & \gamma \equiv 0(M) \\ 0; & \text{otherwise} \end{cases}$$

and $\mathfrak{s}_\alpha(\beta) := \sum_{\kappa=1}^{\beta} \kappa^\alpha$ with $\alpha, \beta \in \mathbb{Z}_+$. From (38) we find that for fixed ν, $0 \le \nu < N$,

$$\alpha_{n-\nu}/\alpha_{n-N} \approx n_0^{-(N-\nu)/M} \Phi_\nu(n_0^{1/M}) \quad (n \to \infty), \tag{39}$$

where $\Phi_\nu(z)$ is the Poincaré series in z^{-1} given by

$$\Phi_\nu(z) = \Phi_\nu^{(P)}(z) := \exp\left(\sum_{\gamma=1}^{\infty} \bar{Q}_\gamma^{(\nu)} z^{-\gamma}\right)$$

with

$$\bar{Q}_\gamma^{(\nu)} = \bar{Q}_\gamma^{(\nu)}(P) := \sum_{0 \le \lambda < \gamma/M} \binom{\gamma/M - 1}{\lambda} \left(\frac{\nu}{Mc_M}\right)^\lambda Q_{\gamma-\lambda M}^{(N-\nu)}(P) + \begin{cases} \frac{(N-\nu)\nu^{\gamma/M}}{(Mc_M)^{\gamma/M}\gamma}; & \gamma \equiv 0(M) \\ 0; & \text{otherwise.} \end{cases}$$

Moreover, by Theorem 2,

$$\alpha_{n-N} \approx \frac{K(P)}{\sqrt{2\pi n}} n_0^{-(n-N)/M} \exp\left(P(n_0^{1/M})\right) \exp\left(\sum_{\gamma=1}^{\infty} \mathscr{B}_\gamma(P,N) n_0^{-\gamma/M}\right) \quad (n \to \infty), \tag{40}$$

where

$$\mathscr{B}_\gamma(P,N) := \sum_{0 \le \lambda < \gamma/M} \binom{\gamma/M - 1}{\lambda} \left(\frac{N}{Mc_M}\right)^\lambda \mathscr{B}_{\gamma-\lambda M}(P) + \begin{cases} \frac{M(\gamma+M-2N)}{2\gamma(\gamma+M)} \left(\frac{N}{Mc_M}\right)^{\gamma/M}; & \gamma \equiv 0(M) \\ 0; & \gamma \equiv i(M),\ 1 \le i < M/2 \\ (-1)^{(\mu_\gamma+\gamma)/M} \binom{\mu_\gamma/M}{(\mu_\gamma+\gamma)/M} \left(\frac{N}{Mc_M}\right)^{(\mu_\gamma+\gamma)/M} c_{\mu_\gamma}; & \gamma \equiv i(M),\ M/2 \le i < M. \end{cases}$$

Here, μ_γ denotes the integer satisfying $\mu_\gamma \equiv -\gamma(M)$ and $1 \le \mu_\gamma \le M$. Combining (37) with (39) and (40) yields part (ii) of the following result; part (i) follows by applying Theorem 1 and Theorem 3(i) in a similar manner.

Theorem 4. *Let $P(z)$ and $Q(z)$ be as above and consider the entire function $Q(z)\exp(P(z)) = \sum_{n=0}^{\infty} \beta_n z^n$.*

(i) *If $P(z)$ meets the conditions ($\mathscr{P}0$) and ($\mathscr{P}3$) then we have*

$$\beta_n \sim \frac{d_N K(P)}{\sqrt{2\pi n}} n_0^{-(n-N)/M} \exp\left(P(n_0^{1/M})\right) \quad (n \to \infty) \tag{41}$$

with $K(P)$ as in Theorem 1.

(ii) *If $M \geq 2$ and $P(z)$ meets the conditions ($\mathscr{P}1$), ($\mathscr{P}2$), and ($\mathscr{P}3$) then there exist constants $\mathscr{C}_\gamma = \mathscr{C}_\gamma(P,Q)$ such that*

(42)

$$\beta_n \approx \frac{d_N K(P)}{\sqrt{2\pi n}} n_0^{-(n-N)/M} \exp\left(P(n_0^{1/M})\right) \left\{1 + \sum_{\gamma=1}^{\infty} \mathscr{C}_\gamma n_0^{-\gamma/M}\right\} \quad (n \to \infty).$$

Moreover, the $\mathscr{C}_\gamma$'s satisfy the identity

(43)

$$1 + \sum_{\gamma=1}^{\infty} \mathscr{C}_\gamma z^{-\gamma} = \exp\left(\sum_{\gamma=1}^{\infty} \mathscr{B}_\gamma(P,N) z^{-\gamma}\right) \left\{1 + \sum_{0 \leq \nu < N} d_\nu d_N^{-1} z^{-(N-\nu)} \Phi_\nu(z)\right\}$$

with $\mathscr{B}_\gamma(P,N)$ and $\Phi_\nu(z)$ as defined above.

5. Applications

5.1. An asymptotic property of the Hermite polynomials

Define a sequence $\widetilde{H}_n(x)$ of integral polynomials in x, $\deg\left(\widetilde{H}_n(x)\right) = n$, by the condition that

$$\sum_{n=0}^{\infty} \frac{\widetilde{H}_n(x)}{n!} z^n = \left(\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n\right)^{-1} = \exp(z^2 - 2xz), \tag{44}$$

where $H_n(x)$ is the n^{th} Hermite polynomial, $H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$. From the definition (44) one deduces the recursion

$$\begin{aligned} \widetilde{H}_n(x) &= -2x\widetilde{H}_{n-1}(x) + 2(n-1)\widetilde{H}_{n-2}(x), \quad n \geq 2 \\ \widetilde{H}_0(x) &= 1, \quad \widetilde{H}_1(x) = -2x. \end{aligned} \tag{45}$$

Comparing (45) with the recursive definition of the Hermite polynomials we find that

$$\widetilde{H}_n(x) = i^n H_n(ix), \quad n \geq 0. \tag{46}$$

On the other hand, for each fixed real number $x<0$, the polynomial $P_x(z):=z^2-2xz$ satisfies the requirements of Theorem 2 and, hence, (26) yields an asymptotic expansion of $\widetilde{H}_n(x)/(n!)$. Combining this with (46) and Stirling's expansion of factorials we then obtain a full asymptotic expansion for the sequence $H_n(ix)$ for each fixed $x<0$.

Proposition 2. *For each fixed real number $x<0$ there exist constants $\eta_\nu=\eta_\nu(x)$ such that as $n\to\infty$*

(47)
$$H_n(ix)\approx\frac{(-1)^n}{\sqrt{2}}i^n(2n)^{n/2}\exp\left(-n/2-x(2n)^{1/2}-x^2/2\right)\left\{1+\sum_{\nu=1}^{\infty}\eta_\nu n^{-\nu/2}\right\}.$$

The η_ν's satisfy the formal identity

$$1+\sum_{\nu=1}^{\infty}\eta_\nu z^{-\nu}=\exp\left(\sum_{\nu=1}^{\infty}\widetilde{\mathscr{B}}_\nu z^{-\nu}\right), \tag{48}$$

where

$$\widetilde{\mathscr{B}}_\nu:=2^{\nu/2}\mathscr{B}_\nu(P_x)+\begin{cases}\frac{4B_{\nu/2+1}}{\nu(\nu+2)}; & \nu\equiv 2(4)\\ 0; & \text{otherwise,}\end{cases}$$

with $B_{\nu/2+1}$ a Bernoulli number and $\mathscr{B}_\nu(P_x)$ as given by (25).

Proposition 2 can be used to determine, for $\alpha=\pm1/2$, the coefficients $C_\nu(\alpha;z)$ in Perron's asymptotic expansion [14, Theorem 8.22.3] for Laguerre polynomials. The result is that, for z in the complex plane cut along the non-negative part of the real axis,

$$C_\nu(-1/2;z)=\left\langle y^{-\nu},\exp\left(\sum_{\nu=1}^{\infty}\widetilde{\mathscr{B}}'_\nu y^{-\nu}\right)\right\rangle,\quad \nu\geq 1$$

$$C_\nu(1/2;z)=\left\langle y^{-\nu},\exp\left(\sum_{\nu=1}^{\infty}\widetilde{\mathscr{B}}''_\nu y^{-\nu}\right)\right\rangle,\quad \nu\geq 1,$$

where

$$\widetilde{\mathscr{B}}'_\nu:=\mathscr{B}_\nu\left(P_{-\sqrt{-z}}\right)-\left(1-2^{-\nu/2}\right)\times\begin{cases}\frac{4B_{\nu/2+1}}{\nu(\nu+2)}; & \nu\equiv 2(4)\\ 0; & \text{otherwise}\end{cases}$$

and

$$\widetilde{\mathscr{B}}''_\nu:=\sum_{0\leq\lambda<\nu/2}\binom{\lambda-\nu/2}{\lambda}2^{-\lambda}\mathscr{B}_{\nu-2\lambda}\left(P_{-\sqrt{-z}}\right)$$

$$+\begin{cases} \binom{1/2}{(\nu+1)/2} 2^{-(\nu-1)/2}\sqrt{-z}; & \nu \text{ odd} \\ 2^{2-\nu/2} \sum\limits_{0\le\lambda<\nu/4} \binom{2\lambda-\nu/2+1}{2\lambda+1} \frac{B_{\nu/2-2\lambda}}{(\nu-4\lambda)(\nu-4\lambda-2)} - \frac{2^{-(\nu-2)/2}}{\nu(\nu+2)}; & \nu\equiv 0(4) \\ 2^{2-\nu/2} \sum\limits_{0\le\lambda<\nu/4} \binom{2\lambda-\nu/2}{2\lambda} \frac{B_{\nu/2-2\lambda+1}}{(\nu-4\lambda)(\nu-4\lambda+2)} + \frac{2^{-(\nu-2)/2}-4B_{\nu/2+1}}{\nu(\nu+2)}; & \nu\equiv 2(4). \end{cases}$$

In particular we find from these formulas that

$$C_1(-1/2;z) = \frac{1}{4\sqrt{-z}}\left(\frac{1}{3}z^2 - z\right)$$

and

$$C_1(1/2;z) = \frac{1}{4\sqrt{-z}}\left(\frac{1}{3}z^2 - 3z\right).$$

For $\alpha=-1/2$ this disagrees with Van Assche's calculation of $C_1(\alpha;z)$ in [15] which also affects some of the results in Section 4 of that paper.

5.2. Counting finite group actions

(a) *Solution of Problem* (1)

For a finite group G of order m let $P_G(z) := \sum_{d|m} \frac{s_G(d)}{d} z^d$, where $s_G(d)$ denotes the number of subgroups of index d in G. By (5) we have

$$|\mathrm{Hom}(G,S_n)| = n!\,\langle z^n, \exp(P_G(z))\rangle, \quad n\ge 0,$$

and if $G\neq 1$ the polynomial $P_G(z)$ satisfies the hypotheses of Theorem 2. Hence, Theorem 2 combined with Stirling's expansion (7) yields the following solution of Problem (1).

Theorem 5. *Let G be a finite group of order $m\ge 2$. Then there exist constants $\widetilde{\mathscr{C}}_\nu = \widetilde{\mathscr{C}}_\nu(G)$ such that the number $|Hom(G,S_n)|$ of G-actions on an n-set has the asymptotic expansion*

$$(49)\qquad |\mathrm{Hom}\,(G,S_n)| \approx K_G n^{(1-1/m)n} \exp\left(-\frac{m-1}{m}n + \sum_{\substack{d|m\\ d<m}} \frac{s_G(d)}{d} n^{d/m}\right) \times \left\{1+\sum_{\nu=1}^{\infty} \widetilde{\mathscr{C}}_\nu(G) n^{-\nu/m}\right\} \quad (n\to\infty),$$

where

$$K_G := \begin{cases} m^{-1/2}; & m \text{ odd} \\ m^{-1/2}\exp\left(-\frac{(s_G(m/2))^2}{2m}\right); & m \text{ even.} \end{cases}$$

Moreover, the $\widetilde{\mathscr{C}}_\nu$*'s satisfy the identity*

(50)
$$1+\sum_{\nu=1}^{\infty}\widetilde{\mathscr{C}}_\nu(G)z^{-\nu}=\exp\left(\sum_{\nu=1}^{\infty}\widetilde{\mathscr{B}}_\nu(G)z^{-\nu}\right),$$

with

$$\widetilde{\mathscr{B}}_\nu(G):=\mathscr{B}_\nu(P_G)+\begin{cases}\frac{m^2B_{\nu/m+1}}{\nu(\nu+m)}; & \nu\equiv m(2m)\\ 0; & \textit{otherwise}\end{cases}$$

and $\mathscr{B}_\nu(P_G)$ *as given by* (25).

Corollary 2. *Let* G *be a finite group of order* m. *Then*
(51)
$$|\mathrm{Hom}\,(G,S_n)|\sim K_G n^{(1-1/m)n}\exp\left(-\frac{m-1}{m}n+\sum_{\substack{d|m\\ d<m}}\frac{s_G(d)}{d}n^{d/m}\right)\ (n\to\infty),$$

where K_G *is defined as in Theorem* 5.

This is trivial for $m=1$ and follows from Theorem 5 otherwise. An alternative approach to formula (51) is provided by Theorem 1.

From (50) we obtain in particular the following values for the coefficients $\widetilde{\mathscr{C}}_1(G), \widetilde{\mathscr{C}}_2(G)$, and $\widetilde{\mathscr{C}}_3(G)$:

(i)	$G=C_2$:	$\widetilde{\mathscr{C}}_1=7/24$,	$\widetilde{\mathscr{C}}_2=-119/1152$,	$\widetilde{\mathscr{C}}_3=-7933/414720$.
(ii)	$G=C_3$:	$\widetilde{\mathscr{C}}_1=-1/6$,	$\widetilde{\mathscr{C}}_2=25/72$,	$\widetilde{\mathscr{C}}_3=-289/1296$.
(iii)	$G=C_4$:	$\widetilde{\mathscr{C}}_1=-1/4$,	$\widetilde{\mathscr{C}}_2=17/96$,	$\widetilde{\mathscr{C}}_3=95/128$.
(iv)	$G=C_2^2$:	$\widetilde{\mathscr{C}}_1=-3/4$,	$\widetilde{\mathscr{C}}_2=47/32$,	$\widetilde{\mathscr{C}}_3=-119/128$.
(v)	$G=C_5$:	$\widetilde{\mathscr{C}}_1=0$,	$\widetilde{\mathscr{C}}_2=0$,	$\widetilde{\mathscr{C}}_3=-1/10$.
(vi)	$G=C_6$:	$\widetilde{\mathscr{C}}_1=-1/6$,	$\widetilde{\mathscr{C}}_2=-17/72$,	$\widetilde{\mathscr{C}}_3=179/1296$.
(vii)	$G=S_3$:	$\widetilde{\mathscr{C}}_1=-1/2$,	$\widetilde{\mathscr{C}}_2=-11/24$,	$\widetilde{\mathscr{C}}_3=59/48$.

(viii) $|G|=m\geq 7$:

$$\widetilde{C}_1(G)=0,$$

$$\widetilde{\mathscr{C}}_2(G)=\begin{cases}-s_G(m/2-2)/m; & m=8,\,12\\ 0; & \text{all other } m\geq 7,\end{cases}$$

$$\widetilde{\mathscr{C}}_3(G)=\begin{cases}-\left(s_G(3)\right)^2/18; & m=9\\ -s_G(m/2)s_G(m/2-3)/m; & m=8,\,10,\,12,\,18\\ 0; & \text{all other } m\geq 7.\end{cases}$$

Remark. The number $|\mathrm{Hom}\,(C_m,S_n)|$ of C_m-actions on an n-set can, of course, be interpreted combinatorially as the number of solutions of the equation $X^m=1$

in the symmetric group S_n. If we denote by $G(m,n)$ the number of elements of (exact) order m in S_n then, by Möbius inversion,

$$G(m,n) = \sum_{d|m} \mu(m/d)|\text{Hom}\,(C_d, S_n)|,$$

and hence, by Corollary 2, in particular

$$G(m,n) = |\text{Hom}\,(C_m, S_n)|\{1 + \mathcal{O}(e^{-n})\} \quad (n \to \infty),$$

where the implied constants may depend on m but not on n. It follows that, for every fixed integer $m \geq 2$, the asymptotic expansion provided by Theorem 5 for $|\text{Hom}(C_m, S_n)|$ applies without change to the function $G(m,n)$ too.

(b) *Asymptotic stability of finite groups*

A noteworthy consequence of Corollary 1 is that the asymptotic behaviour of the function $|\text{Hom}\,(G, S_n)|$ attached to a finite group G determines the function itself. More precisely, formula (5) and Corollary 1 imply the following.

Proposition 3. *Let G and H be two finite groups. Then the following statements are equivalent:*
(i) $|G| = |H| =: m$ *and* $s_G(d) = s_H(d)$ *for all* $d \mid m$.
(ii) $|\text{Hom}(G, S_n)| = |\text{Hom}(H, S_n)|$ *for all* $n \geq 0$.
(iii) $|\text{Hom}(G, S_n)| \sim |\text{Hom}(H, S_n)|$ $(n \to \infty)$.

The question as to what kind of information on the structure of a finite group G is contained in the order $m = |G|$ and the subgroup numbers $s_G(d)$ for $d \mid m$ and to what extent G is determined by these data might deserve some further investigation. The existence of non-isomorphic p-groups with isomorphic subgroup lattices shows on the one hand that G will not be determined up to isomorphism; on the other hand these data allow to decide for example whether G is soluble respectively nilpotent.

(c) *On a conjecture of Chowla, Herstein, and Moore*

As in (2) denote by T_n the number of C_2-actions on a set of cardinality n. In [2] Chowla et al. showed among other things that the quotient T_n/T_{n-1} is asymptotically equal to $n^{1/2}$, [2, Theorem 3], and conjectured that, more precisely, an asymptotic expansion of the form

$$T_n/T_{n-1} \approx n^{1/2} \left\{ 1 + \sum_{\nu=1}^{\infty} \widetilde{Q}_\nu n^{-\nu/2} \right\} \quad (n \to \infty) \tag{52}$$

holds with appropriate constants $\widetilde{Q}_\nu$. That this is indeed the case was first established by Moser and Wyman, cf. [11, 2.12]; they observed that

$$T_n = \frac{H_n(i/\sqrt{2})}{i^n 2^{n/2}},$$

where i is the imaginary unit and H_n denotes, again, the n^{th} Hermite polynomial, and then used a classical formula of generalized Hilb's type for Hermite polynomials ([14, Theorem 8.22.7]) to prove (52), obtaining the first two coefficients explicitly as $\widetilde{Q}_1 = 1/2$ and $\widetilde{Q}_2 = -1/8$. Later, the conjecture was extended to cyclic groups to the effect that the quotient $|\text{Hom}\,(C_m, S_n)|/|\text{Hom}\,(C_m, S_{n-1})|$ should have an asymptotic expansion for fixed m as $n \to \infty$ of the form $n^{1-1/m}$ times a Poincaré series in $n^{-1/m}$ whose coefficients depend only on m; but as far as we are aware no further progress was made. However, such a statement is indeed true, even for arbitrary finite groups. As a consequence of Theorem 3 we have the following.

Theorem 6. *Let G be a finite group of order $m \geq 2$. Then there exist constants $\widetilde{Q}_\nu = \widetilde{Q}_\nu(G)$ such that as $n \to \infty$*

$$|\text{Hom}\,(G, S_n)|/|\text{Hom}\,(G, S_{n-1})| \approx n^{1-1/m} \left\{ 1 + \sum_{\nu=1}^{\infty} \widetilde{Q}_\nu(G) n^{-\nu/m} \right\}. \tag{53}$$

Moreover, the $\widetilde{Q}_\nu$'s satisfy the formal identity

$$1 + \sum_{\nu=1}^{\infty} \widetilde{Q}_\nu(G) z^{-\nu} = \exp\left(\sum_{\nu=1}^{\infty} Q_\nu(G) z^{-\nu} \right), \tag{54}$$

where

$$Q_\nu = Q_\nu(G) := -\sum_{1 \leq \lambda < \nu/m} \binom{\nu/m - 1}{\lambda} \mathscr{B}_{\nu - m\lambda}(P_G)$$
$$+ \begin{cases} -\frac{m(\nu+m-2)}{2\nu(\nu+m)}; & \nu \equiv 0(m) \\ (-1)^{(\nu-m+d)/m} \binom{d/m}{(\nu+d)/m} \frac{s_G(d)}{d}; & \nu \equiv m - d(m),\ d \mid m,\ d < m \\ 0; & \textit{otherwise.} \end{cases}$$

From (54) we obtain in particular the following values for the coefficients $\widetilde{Q}_1(G), \ldots, \widetilde{Q}_{m+3}(G)$:

$$\widetilde{Q}_\nu(G) = \begin{cases} \frac{s_G(d)}{m}; & \nu = m - d,\ d \mid m,\ d < m \\ 0; & \text{all other } \nu < m, \end{cases}$$

$$\widetilde{Q}_m(G) = \begin{cases} -\frac{m-1}{2m}; & m \text{ odd} \\ -\frac{m-1}{2m} + \frac{1}{2}\left(\frac{s_G(m/2)}{m}\right)^2; & m \text{ even,} \end{cases}$$

$$\widetilde{Q}_{m+1}(G) = \begin{cases} -1/8; & G = C_2 \\ 1/9; & G = C_3 \\ 1/8; & G = C_4 \\ 3/8; & G = C_2^2 \\ 1/18; & G = C_6 \\ 1/6; & G = S_3 \\ 0; & \text{all other groups } G, \end{cases}$$

$$\widetilde{Q}_{m+2}(G) = \begin{cases} 7/128; & G = C_2 \\ -2/9; & G = C_3 \\ -9/128; & G = C_4 \\ -75/128; & G = C_2^2 \\ 1/8; & G = C_6 \\ 7/24; & G = S_3 \\ m^{-2} s_G(m/2-2)\,(2 + s_G(m/2))\,; & m = 8,\ 12 \\ 0; & \text{all other cases,} \end{cases}$$

and

$$\widetilde{Q}_{m+3}(G) = \begin{cases} -1/32; & G = C_2 \\ 11/81; & G = C_3 \\ -9/16; & G = C_4 \\ 3/16; & G = C_2^2 \\ 2/25; & G = C_5 \\ -31/648; & G = C_6 \\ -37/72; & G = S_3 \\ (2/m)^2 s_G(m/2) s_G(m/2-3); & m = 8,\ 10,\ 12,\ 18 \\ 2(s_G(3)/9)^2; & m = 9 \\ 0; & \text{all other cases.} \end{cases}$$

As mentioned in the introduction Theorems 5 and 6, apart from their intrinsic interest, also lead to a determination of the subgroup growth for a large class of virtually free groups. This is explained in [9] and [10].

References

[1] A. BRAUER: On a problem of partitions, *Amer. J. Math.* **64** (1942), 299–312.

[2] S. CHOWLA, I. N. HERSTEIN, and W. K. MOORE: On recursions connected with symmetric groups I, *Can. J. Math.* **3** (1951), 328–334.

[3] S. CHOWLA, I. N. HERSTEIN, and W. R. SCOTT: The solutions of $x^d = 1$ in symmetric groups, *Norske Vid. Selsk.* **25** (1952), 29–31.

[4] N. G. DE BRUIJN: *Asymptotic Methods in Analysis*, Dover Publ., New York, 1981.

[5] A. DRESS and T. MÜLLER: Decomposable functors and the exponential principle, *Adv. in Math.* **129** (1997), 188–221.

[6] W. K. HAYMAN: A generalization of Stirling's formula, *J. Reine Angew. Math.* **196** (1956), 67–95.

[7] B. HARRIS and L. SCHOENFELD: The number of idempotent elements in symmetric semigroups, *J. Comb. Theory* **3** (1967), 122–135.

[8] B. HARRIS and L. SCHOENFELD: Asymptotic expansion for the coefficients of analytic functions, *Illinois J. Math.* **12** (1968), 264–277.

[9] T. MÜLLER: Subgroup growth of free products, *Invent. Math.* **126** (1996), 111–131.

[10] T. MÜLLER: Subgroup growth of cyclic covers, preprint.

[11] L. MOSER and M. WYMAN: On solutions of $x^d = 1$ in symmetric groups, *Can. J. Math.* **7** (1955), 159–168.

[12] L. MOSER and M. WYMAN: Asymptotic expansions, *Can. J. Math.* **8** (1956), 225–233.

[13] G. PÓLYA: Über die Nullstellen sukzessiver Derivierten, *Math. Z.* **12** (1922), 36–60.

[14] G. SZEGŐ: *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Vol. XXIII, Providence, Rhode Island, 1967.

[15] W. VAN ASSCHE: Weighted zero distribution for polynomials orthogonal on an infinite interval, *Siam J. Math. Anal.* **16** (1985), 1317–1334.

[16] H. S. WILF: The asymptotics of $e^{P(z)}$ and the number of elements of each order in S_n, *Bull. Amer. Math. Soc.* **15** (1986), 228–232.

[17] E. T. WHITTAKER and G. N. WATSON: *A Course of Modern Analysis*, Cambridge, 1958.

Thomas Müller

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501, Bielefeld
Germany
thomue@mathematik.uni-bielefeld.de